# Future and Present Values of Annuities

Finite Math

28 February 2019

# Quiz

What is an annuity?

# Future Value

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*Note that the payments are made at the end of each period.*

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## Example

*What is the value of an annuity at the end of* $10$ *years if* $\$1,000$ *is deposited every* $3$ *months into an account earning* $8\%$ *compounded quarterly. How much of this value is interest?*

# Now You Try It!

## Example

*If* $1,000 *is deposited at the end of each year for* 5 *years into an ordinary annuity earning* 8.32% *compounded annually, what will be the value of the annuity at the end of the* 5 *years?*

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## Solution

$\$5,904.15$

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**Example**

*New parents are trying to save for their child's college and want to save up* $\$80,000$ *in* $17$ *years. They have found an account that will pay* $8\%$ *interest compounded quarterly. How much will they have to deposit every quarter in order to have a value of* $\$80,000$*?*

# Now You Try It!

## Example

*A bond issue is approved for building a marina in a city. The city is required to make regular payments every* 3 *months into a sinking fund paying* 5.4% *compounded quarterly. At the end of* 10 *years, the bond obligation will be retired with a cost of* $5,000,000. *How much will the city have to pay each quarter?*

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**Solution**

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We will look at making a large deposit in order to have a fund which we can make constant withdraws from. We make an initial deposit, then make withdraws at the end of each interest period. We should have a balance of $0 at the end of the predetermined amount of time the fund should last.

# Example

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*How much should you deposit into an account paying* 6% *compounded semiannually in order to be able to withdraw* \$2000 *every* 6 *months for* 2 *years? (At the end of the* 2 *years, there should be a balance of* \$0 *in the account.)*

# Solution

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So adding up the present values of all these will give us the amount of money we should deposit into the account now

$$D = \$2000(1.03)^{-1} + \$2000(1.03)^{-2} + \$2000(1.03)^{-3} + \$2000(1.03)^{-4} = \$7434.20$$

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## Solution

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# Amortization

*Amortization* is the process of paying off a debt. The formula for present value of an annuity will allow us to model the process of paying off a loan or other debt.

# Amortization

## Example

*Suppose you take out a* 5*-year,* $25,000 *loan from your bank to purchase a new car. If your bank gives you* 1.9% *interest compounded monthly on the loan and you make equal monthly payments, how much will your monthly payment be?*

# Now You Try It!

## Example

*If you sell your car to someone for* $\$2,400$ *and agree to finance it at* $1\%$ *per month on the unpaid balance, how much should you receive each month to amortize the loan in* $24$ *months? How much interest will you receive?*

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$PMT = \$112.98$, $I = \$311.52$

# Combination Example

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## Example

*The full retirement age in the US is* 67 *for people born in* 1960 *or later. Suppose you start saving for retirement at* 27 *years old and you would like to save enough to withdraw* \$40,000 *per year for the next* 20 *years. If you find a retirement savings account (for example, a Roth IRA) which pays* 4% *interest compounded annually, how much will you have to deposit per year from age* 27 *until you retire in order to be able to make your desired withdraws?*

# Now You Try It!

## Example

*Lincoln Benefit Life offered an ordinary annuity earning* $6.5\%$ *compounded annually. If* $\$2,000$ *is deposited annually for the first* $25$ *years, how much can be withdrawn annually for the next* $20$ *years?*

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## Solution

$\$10,688.87$